Yet Another Proof of Steiner-Lehmus Theorem

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The Steiner-Lehmus theorem reads as follows: If the bisectors of the two internal angles of a triangle are equal, then the triangle is isosceles (see Figure 1). In fact, the theorem is the converse of a well-known theorem that the angle bisectors of an isosceles triangle are congruent, a direct result of the Angle-Side-Angle congruency proved in the *Elements* [1], Book I. As noted in [2], "in the year 1840 Professor Lehmus from Berlin asked the famous Swiss geometer Jacob Steiner for a proof of" the above mentioned theorem. "Steiner soon found a proof but did not publish it until 1844. In 1850 Lehmus found a proof of his own. However it was the French mathematician Rougevain who was the first to publish a proof in 1842." There are more than 80 accepted proofs for it today [3]. For an analysis of some proofs for the theorem see [2, 4, 5] and for a recent proof see [6].

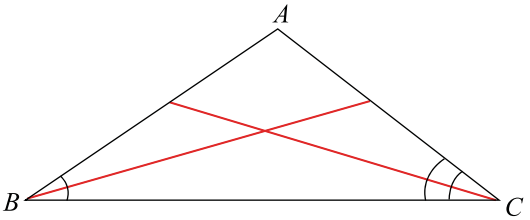

Figure 1 The Steiner-Lehmus theorem.

Around 1850 in England the problem was posed with the additional requirement that the proof be direct as all the proofs up to then had been by *reductio ad absurdum*. Allegedly "direct" proofs have been published, but not everyone agrees that these proofs are "direct". J. J. Sylvester in 1852 asserted that there can be no direct proof of the Steiner-Lehmus theorem. We may agree with McBride, in his historical account of the theorem up to 1940 [7], in that a proof is rightly discredited as being a "direct" proof if it depends on indirectly proved lemmas. McBride concluded that "If it is held, as I hold, that Euclid I. 14, Euclid I. 29, Euclid I. 32, and the theorem of Pythagoras have no direct proof, then the Bisectors Theorem has not been proved directly, nor is it likely to be." Conway and Rybain in [8] claimed that this old query is no longer the right question to ask, remarking that there is no agreed meaning to this term; see also [9]. However, as noted in [10], the question of "direct proof" can be formalized. In fact, a precise definition of a "direct proof" inside both classical and intuitionistic logic has been provided by Pambuccian [11], who proved, without presenting the direct proofs,

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that direct proofs must exist in both the classical logic and the intuitionistic logic setting (using the intuitionistic axiom system for Euclidean geometry proposed in [12]).

In this note, a proof of this famous theorem will be given. Our proof relies on, among other things, the fact that the angles of any triangle sum to 180, which is proved in the *Elements* by indirect proof. Our proof is elementary and we have found no similar such proof in the literature.

We begin with a very simple lemma.

Lemma. *Let O be an arbitrary point on the bisector of the angle $\angle G$. In Figure 2, if $\overline{OM} = \overline{ON}$ then $\angle O_1 = \angle O_2$.*

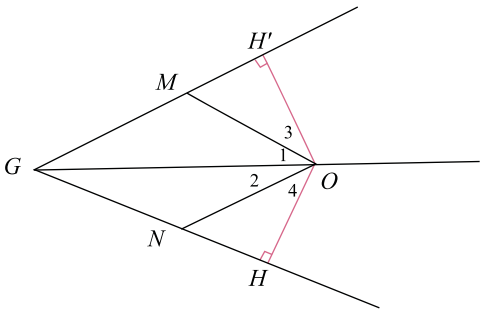

Figure 2 Diagram for proof of Lemma 2.

Proof. The two right triangles $\triangle GOH'$ and $\triangle GOH$ are congruent (by the hypotenuse-angle theorem). Thus, $\angle O_1 + \angle O_3 = \angle O_2 + \angle O_4$. Also, two right triangles $\triangle MOH'$ and $\triangle NOH$ are congruent (by the hypotenuse-leg theorem), and we have $\angle O_3 = \angle O_4$. Hence, $\angle O_1 = \angle O_2$. ■

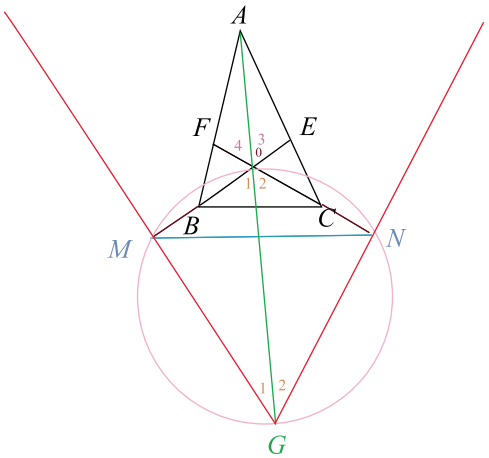

Figure 3 Diagram for proof of the Steiner-Lehmus theorem.

Proof of the Steiner-Lehmus theorem: As in Figure 3, let ABC be a triangle, and CF and BE be the bisectors of the internal angles $\angle C$ and $\angle B$, respectively, such that $\overline{BE} = \overline{CF}$ and BE meets CF in O. Extend BE and CF beyond the vertices B and C

to the points M and N, respectively, such that $\overline{BM} = \overline{OE}$ and $\overline{CN} = \overline{OF}$. Then, by assumption, $\overline{OM} = \overline{ON}$.

Let the straight line AO meet the circumcircle of the three points O, M, and N in a point G. Noting that $\angle G_1 = \angle G_2$ (since these arcs are the inscribed angles of two arcs of equal length), it follows, by the lemma, that $\angle O_1 = \angle O_2$, and $\angle O_3 = \angle O_4$. Thus, $\triangle AFO$ and $\triangle AEO$ are congruent (AAS); note that AO is the bisector of $\angle A$, this is because of the concurrency of angle bisectors. Hence $\overline{AE} = \overline{AF}$. Also, since $\triangle FBO$ and $\triangle ECO$ are congruent (ASA), we have $\overline{EC} = \overline{FB}$, which completes the proof.

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Summary. In this note, we give a proof for the famous Steiner-Lehmus Theorem.

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